TATE COHOMOLOGY OVER FAIRLY GENERAL RINGS

PETER JØRGENSEN

Abstract. Tate cohomology was originally defined over finite groups. More recently, Avramov and Martsinkovsky showed how to extend the definition so that it now works well over Gorenstein rings.

This paper improves the theory further by giving a new definition that works over more general rings, specifically, those with a dualizing complex.

The new definition of Tate cohomology retains the desirable properties established by Avramov and Martsinkovsky. Notably, there is a long exact sequence connecting it to ordinary Ext groups.

0. Introduction

Tate cohomology was originally defined over finite groups, and has been used to great effect in group representation theory.

More recently, Avramov and Martsinkovsky accomplished in [1] an extension of the definition so that it now works well over Gorenstein rings. In fact, [1] went so far as to define Tate Ext groups,

$$\widehat{\operatorname{Ext}}_A^i(M,N), \tag{1}$$

which have classical Tate cohomology as the special case $\widehat{\operatorname{Ext}}_{\mathbb{Z}G}^i(\mathbb{Z},N)$.

Moreover, a new key result was introduced in [1]: Tate and ordinary Ext groups fit into a long exact sequence

$$0 \to \operatorname{Ext}^1_{\mathcal{G}}(M,N) \longrightarrow \operatorname{Ext}^1(M,N) \longrightarrow \widehat{\operatorname{Ext}}^1(M,N) \longrightarrow \cdots , \tag{2}$$

where the $\operatorname{Ext}_{\mathcal{G}}$'s are relative Ext groups defined by means of what is known as (proper) Gorenstein projective resolutions in the first variable. This illuminates the connection between Tate cohomology and ordinary Ext groups, and also explains why "Gorenstein phenomena" play an important role in Tate cohomology theory.

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To be more precise about [1], what it really did was to construct the Tate Ext groups from equation (1) over any noetherian ring, but only when M is a finitely generated module of finite Gorenstein projective dimension. So to have the Tate Ext groups defined everywhere, it is necessary to require that each finitely generated module has finite Gorenstein projective dimension. In turn, this is the same as requiring the ring to be Gorenstein. Hence, for non-Gorenstein rings there are modules where the Tate Ext groups of [1] remain undefined.

The present paper improves the theory by giving a new definition of Tate Ext groups that works for all modules, including the ones with infinite Gorenstein projective dimension. Moreover, it is proved that the new definition coincides with the definition from [1] when they both work, and that the new definition retains the desirable properties established in [1], notably the long exact sequence (2).

The results apply over fairly general rings, whence the title of the paper. For instance, noetherian commutative rings with dualizing complexes are covered, and so too is a large class of non-commutative rings. See remark 1.3 for more information.

Let me close the introduction with a synopsis of the paper.

Section 1 gives the setup which is used in the rest of the paper.

Section 2 gives the new definition of Tate Ext groups (definition 2.2), proves that short exact sequences of modules result in long exact sequences of Tate Ext groups (proposition 2.3), recalls the definition of Tate Ext groups which was given in [1] (remark 2.5), and proves that the definitions coincide when they both work (proposition 2.7).

Sections 3 and 4 form an interlude. Section 3 just contains some lemmas. Section 4 considers Gorenstein projective modules, and shows how the machinery of the paper gives rise to Gorenstein projective resolutions (lemma 4.5).

Section 5 proves that the new definition of Tate Ext groups fits into the long exact sequence (2) (theorem 5.3).

As a spin off along the way, it is proved that each module M admits a Gorenstein projective precover $G \longrightarrow M$ whose kernel Z is particularly nice. Namely, Z has the property that each projective resolution of Z is even a (proper) Gorenstein projective resolution of Z (theorem 4.7). This generalizes previous results by several authors (see remark 4.6).

1. Setup

Definition 1.1. Let A be a ring. By $\mathsf{E}(A)$ is denoted the class of complexes E of A-left-modules so that E consists of projective modules,

is exact, and has $\mathrm{Hom}_A(E, Q)$ exact for each projective A-left-module Q.

I will view $\mathsf{E}(A)$ as a full subcategory of $\mathsf{K}(\mathsf{Pro}\, A)$, the homotopy category of complexes of projective A-left-modules.

Note that $\mathsf{E}(A)$ consists precisely of the complexes known as complete projective resolutions.

Setup 1.2. Throughout this paper, A is a ring for which the inclusion functor

$$e_* : \mathsf{E}(A) \longrightarrow \mathsf{K}(\mathsf{Pro}\, A)$$

has a right-adjoint

$$e^! : \mathsf{K}(\mathsf{Pro}\, A) \longrightarrow \mathsf{E}(A).$$

Remark 1.3. The right-adjoint $e^!$ exists for fairly general rings, whence the title of the paper.

Namely, by [4, prop. 2.2] the right-adjoint $e^!$ exists when A is a noetherian commutative ring with a dualizing complex.

Also, by [4, sec. 4] the right-adjoint $e^!$ exists when A is a left-coherent and right-noetherian k-algebra over the field k for which there exists a left-noetherian k-algebra B and a dualizing complex ${}_BD_A$.

These cases cover many rings arising in practice.

Remark 1.4. If P is a complex of projective modules, then $e^!P$ can be thought of as the best approximation to P by a complete projective resolution.

Elaborating on this, if M is a module with projective resolution P, then $e^!P$ can be thought of as the best approximation to M by a complete projective resolution. This point will be made more precise in lemma 2.6.

2. Tate Ext groups

This section gives the new definition of Tate Ext groups (definition 2.2), and proves that short exact sequences of modules result in long exact sequences of Tate Ext groups (proposition 2.3).

The rest of the section is devoted to recalling the earlier definition of Tate Ext groups which was given in [1] (remark 2.5), and proving that the new and the earlier definition of Tate Ext groups coincide when they both work (proposition 2.7).

Remark 2.1. It is classical that the category of A-left-modules $\mathsf{Mod}(A)$ is equivalent to the full subcategory of $\mathsf{K}(\mathsf{Pro}\, A)$ consisting of projective resolutions of A-left-modules. Let

$$\mathsf{Mod}(A) \xrightarrow{\mathrm{res}} \mathsf{K}(\mathsf{Pro}\, A)$$

be a functor implementing the equivalence.

Definition 2.2. For A-left-modules M and N, the i'th Tate Ext group is

$$\widehat{\mathrm{Ext}}^i(M,N) = \mathrm{H}^i\mathrm{Hom}_A(e^! \operatorname{res} M, N).$$

This is the definition one would expect: As pointed out in remark 1.4, the complex $e^! \operatorname{res} M$ can be thought of as the best approximation to M by a complete projective resolution. So taking Hom into N and taking cohomology should be the way to get Tate Ext groups.

Proposition 2.3. *If*

$$0 \to M' \longrightarrow M \longrightarrow M'' \to 0$$

and

$$0 \to N' \longrightarrow N \longrightarrow N'' \to 0$$

are short exact sequences of A-left-modules, then there are natural long exact sequences

$$\cdots \longrightarrow \widehat{\mathrm{Ext}}^i(M'',N) \longrightarrow \widehat{\mathrm{Ext}}^i(M,N) \longrightarrow \widehat{\mathrm{Ext}}^i(M',N) \longrightarrow \cdots$$

and

$$\cdots \longrightarrow \widehat{\mathrm{Ext}}^i(M,N') \longrightarrow \widehat{\mathrm{Ext}}^i(M,N) \longrightarrow \widehat{\mathrm{Ext}}^i(M,N'') \longrightarrow \cdots .$$

Proof. It is well known that the first short exact sequence in the proposition results in a distinguished triangle in $\mathsf{K}(\mathsf{Pro}\,A)$,

$$\operatorname{res} M' \longrightarrow \operatorname{res} M \longrightarrow \operatorname{res} M'' \longrightarrow .$$

Since e_* is a triangulated functor, so is its adjoint $e^!$, so there is also a distinguished triangle in $\mathsf{E}(A)$,

$$e^! \operatorname{res} M' \longrightarrow e^! \operatorname{res} M \longrightarrow e^! \operatorname{res} M'' \longrightarrow .$$

This again results in a distinguished triangle

$$\mathrm{Hom}_A(e^! \operatorname{res} M'',N) \longrightarrow \mathrm{Hom}_A(e^! \operatorname{res} M,N) \longrightarrow \mathrm{Hom}_A(e^! \operatorname{res} M',N) \longrightarrow$$

whose cohomology long exact sequence is the first long exact sequence in the proposition.

The complex $e^! \operatorname{res} M$ is in $\mathsf{E}(A)$ so consists of projective modules, so the second short exact sequence in the proposition gives a short exact sequence of complexes

$$0 \to \mathrm{Hom}_A(e^! \operatorname{res} M,N') \longrightarrow \mathrm{Hom}_A(e^! \operatorname{res} M,N) \longrightarrow \mathrm{Hom}_A(e^! \operatorname{res} M,N'') \to 0$$

whose cohomology long exact sequence is the second long exact sequence in the proposition. $\square$

Construction 2.4. If P is a complex of A-left-modules, then for each i there is a chain map

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & P^i \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & P^i \arrow[r] \arrow[d, "\partial_P^i"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P^{i-1} \arrow[r, "\partial_P^{i-1}"'] & P^i \arrow[r, "\partial_P^i"'] & P^{i+1} \arrow[r, "\partial_P^{i+1}"'] & P^{i+2} \arrow[r] & \cdots
\end{tikzcd}$$

where the upper complex is null homotopic.

This is useful because I can add the upper complex to T in any chain map $T \xrightarrow{t} P$, and thereby change t so that the i'th component t^i becomes surjective. Doing so does not change the isomorphism class of t in $\mathsf{K}(A)$, the homotopy category of complexes of A-left-modules.

Remark 2.5. The earlier definition of Tate Ext groups which was given in [1] is

$$\widehat{\mathrm{Ext}}^i(M, N) = \mathrm{H}^i\mathrm{Hom}_A(T, N)$$

where T is a complete projective resolution of the A-left-module M. This means that T is in $\mathsf{E}(A)$, consists of finitely generated modules, and sits in a diagram of chain maps

$$T \xrightarrow{t} P \longrightarrow M \tag{3}$$

where $P \longrightarrow M$ is a projective resolution and where t^i is bijective for $i \ll 0$.

Note that not all A-left-modules have complete projective resolutions. In fact, if A is left-noetherian, then the ones that do are exactly the ones which are finitely generated and have finite Gorenstein projective dimension by [1, thm. 3.1].

Lemma 2.6. *Let M be an A-left-module which has a projective resolution P and a complete projective resolution T. Then*

$$e^! P \cong T$$

in $\mathsf{K}(\mathsf{Pro}\, A)$.

Proof. All projective resolutions of M are isomorphic in $\mathsf{K}(\mathsf{Pro}\, A)$, so I may as well prove the lemma for the specific projective resolution P from equation (3).

By applying construction 2.4 to the chain map $T \xrightarrow{t} P$ in cohomological degrees larger than some number, I can assume that t is

surjective. Hence there is a short exact sequence of complexes

$$0 \to K \longrightarrow T \xrightarrow{t} P \to 0. \tag{4}$$

Since both T and P consist of projective modules, the sequence is semi-split and K also consists of projective modules. Moreover, by assumption, t^i is bijective for $i \ll 0$, so $K^i = 0$ for $i \ll 0$. So K is a left-bounded complex of projective modules.

Now let E be in $\mathsf{E}(A)$. In particular, $\mathrm{Hom}_A(E, Q)$ is exact when Q is a projective module. It is classical that $\mathrm{Hom}_A(E, K)$ is then also exact, because K is a left-bounded complex of projective modules. Indeed, this follows by an argument analogous to the one which shows that if X is an exact complex and I is a left-bounded complex of injective modules, then $\mathrm{Hom}_A(X, I)$ is exact.

Since the sequence (4) is semi-split, it stays exact under the functor $\mathrm{Hom}_A(E, -)$. So there is a short exact sequence of complexes

$$0 \to \mathrm{Hom}_A(E, K) \longrightarrow \mathrm{Hom}_A(E, T) \longrightarrow \mathrm{Hom}_A(E, P) \to 0.$$

Since $\mathrm{Hom}_A(E, K)$ is exact, the cohomology long exact sequence shows that there is an isomorphism

$$\mathrm{H}^0\mathrm{Hom}_A(E, T) \cong \mathrm{H}^0\mathrm{Hom}_A(E, P)$$

which is natural in E. That is, there is a natural isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(E, T) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(E, P)$$

which can also be written

$$\mathrm{Hom}_{\mathsf{E}(A)}(E, T) \cong \mathrm{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(E, P)$$

because E and T are in $\mathsf{E}(A)$.

On the other hand, I also have a natural isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(E, P) = \mathrm{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(e_* E, P) \cong \mathrm{Hom}_{\mathsf{E}(A)}(E, e^! P).$$

Combining the last two equations gives a natural isomorphism

$$\mathrm{Hom}_{\mathsf{E}(A)}(E, T) \cong \mathrm{Hom}_{\mathsf{E}(A)}(E, e^! P),$$

proving $T \cong e^! P$ as desired. □

Proposition 2.7. *Let M be an A-left-module which has a complete projective resolution T. Then the new Tate* Ext *groups of this paper (definition 2.2) coincide with the Tate* Ext *groups which were defined in* [1] *(remark 2.5).*

Proof. Lemma 2.6 gives that the projective resolution $\mathrm{res}\, M$ of M satisfies $e^! \,\mathrm{res}\, M \cong T$. Combining this with the formulae in definition 2.2 and remark 2.5 proves the proposition. □

3. Some lemmas

This section collects three lemmas needed later in the paper.

Lemma 3.1. *Let P be in $\mathsf{K}(\mathsf{Pro}\,A)$, and consider the counit morphism*

$$e_*e^!P \xrightarrow{\epsilon_P} P.$$

Let E be in $\mathsf{E}(A)$. Then the induced map

$$\operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(E, e_*e^!P) \xrightarrow{\operatorname{Hom}(E,\epsilon_P)} \operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(E, P)$$

is an isomorphism.

Proof. There is a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathsf{E}(A)}(E, e^!P) \arrow[r, "e_*(-)"] \arrow[dr] & \operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(e_*E, e_*e^!P) \arrow[d, "\operatorname{Hom}(e_*E,\epsilon_P)"] \\
 & \operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(e_*E, P).
\end{tikzcd}$$

The diagonal map is the adjunction isomorphism, and the horizontal map is an isomorphism because e_* is the inclusion functor of a full subcategory.

Hence the vertical map must be an isomorphism, but this is the induced map from the lemma in slightly different notation. $\square$

Remark 3.2. For the following two lemmas, recall that an A-left-module is called Gorenstein projective if it has the form

$$G = \operatorname{Ker}(E^1 \longrightarrow E^2)$$

for some E in $\mathsf{E}(A)$.

A homomorphism $K \xrightarrow{s} N$ is called a relative epimorphism with respect to the class of Gorenstein projective modules if each homomorphism $G \xrightarrow{g} N$ with G Gorenstein projective lifts through s,

$$\begin{tikzcd}
 & K \arrow[d, "s"] \\
G \arrow[ur, dotted] \arrow[r, "g"'] & N.
\end{tikzcd}$$

Lemma 3.3. *Let K be a complex of projective A-left-modules satisfying $\operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(E, K) = 0$ for each E in $\mathsf{E}(A)$, and suppose that*

$$\cdots \longrightarrow K^{i-1} \longrightarrow K^i \xrightarrow{s} N \to 0$$

is exact.

Then $K^i \xrightarrow{s} N$ is a relative epimorphism with respect to the class of Gorenstein projective modules.

Proof. Let G be a Gorenstein projective module, and let $G \xrightarrow{g} N$ be a homomorphism.

By shifting, I can clearly pick a complex E in $\mathsf{E}(A)$ with $G = \operatorname{Ker}(E^{i+1} \longrightarrow E^{i+2})$, and it is not hard to see that there is a chain map $E \xrightarrow{e} K$ which fits together with $G \xrightarrow{g} N$ in a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & E^{i-1} \arrow[rr] \arrow[dd, "e^{i-1}"'] & & E^i \arrow[rr] \arrow[dr] \arrow[dd, "e^i"'] & & E^{i+1} \arrow[rr] \arrow[dd, "e^{i+1}"] & & E^{i+2} \arrow[r] \arrow[dd, "e^{i+2}"] & \cdots \\
& & & & G \arrow[ur, "\ell"'] \arrow[dd, "g"'] & & & & \\
\cdots \arrow[r] & K^{i-1} \arrow[rr] & & K^i \arrow[rr] \arrow[dr, "s"'] & & K^{i+1} \arrow[rr] & & K^{i+2} \arrow[r] & \cdots. \\
& & & & N \arrow[ur] & & & &
\end{tikzcd}$$

Since I have assumed $\operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(E, K) = 0$ for E in $\mathsf{E}(A)$, the chain map e must be null homotopic.

Let h be a null homotopy with $e = h\partial^E + \partial^K h$, consisting of components $E^j \xrightarrow{h^j} K^{j-1}$. Then it is straightforward to prove

$$s \circ (h^{i+1}\ell) = g,$$

so $G \xrightarrow{g} N$ has been lifted through $K^i \xrightarrow{s} N$ as desired. □

Lemma 3.4. *Let K be a complex of projective A-left-modules satisfying $\operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\,A)}(E, K) = 0$ for each E in $\mathsf{E}(A)$, and suppose that*

$$\cdots \longrightarrow K^{i-2} \longrightarrow K^{i-1} \longrightarrow \operatorname{Ker} \partial_K^i \xrightarrow{t} N \to 0$$

is exact.

Then $\operatorname{Ker} \partial_K^i \xrightarrow{t} N$ is a relative epimorphism with respect to the class of Gorenstein projective modules.

Proof. From the data given I can construct a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & K^{i-1} \arrow[rr] \arrow[dr] & & K^i \arrow[rr] \arrow[dr, "s"] & & K^{i+1} \arrow[r] & \cdots. \\
& & \operatorname{Ker} \partial_K^i \arrow[ur] \arrow[dr, "t"'] & & \operatorname{Coker} \partial_K^{i-1} \arrow[ur] & & \\
& & & N \arrow[ur] & & &
\end{tikzcd}$$

It follows from lemma 3.3 that $K^i \xrightarrow{s} \operatorname{Coker} \partial_K^{i-1}$ is a relative epimorphism with respect to the class of Gorenstein projective modules, and it is a small diagram exercise to see that this implies the same for $\operatorname{Ker} \partial_K^i \xrightarrow{t} N$. □

4. Gorenstein projective modules

Let M be an A-left-module with projective resolution P. This section considers the kernel K of the counit morphism $e_* e^! P \xrightarrow{\epsilon_P} P$ (construction 4.1) and shows how it gives rise to a Gorenstein projective resolution of M (lemma 4.5).

This resolution is constructed to be used in the next section. However, as a spin off I also use it to prove (theorem 4.7) that each module M admits a Gorenstein projective precover $G \longrightarrow M$ whose kernel Z is particularly nice. Namely, Z has the property that each projective resolution of Z is even a Gorenstein projective resolution of Z. This generalizes previous results by several authors (see remark 4.6).

Construction 4.1. Let M be an A-left-module with projective resolution P concentrated in non-positive cohomological degrees. By applying construction 2.4 in each degree, I can assume that the counit morphism $e_* e^! P \xrightarrow{\epsilon_P} P$ in $\mathsf{K}(\mathsf{Pro}\, A)$ is represented by a surjective chain map, so setting

$$F = e_* e^! P,$$

there is a short exact sequence of complexes

$$0 \to K \longrightarrow F \longrightarrow P \to 0.$$

Note that since both F and P consist of projective modules, the sequence is semi-split and K also consists of projective modules.

Lemma 4.2. *Consider the situation of construction 4.1. The complex K satisfies*

$$\operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\, A)}(E, K) = 0$$

for E in $\mathsf{E}(A)$.

Proof. The short exact sequence from construction 4.1 is semi-split and therefore gives a distinguished triangle

$$K \longrightarrow F \longrightarrow P \longrightarrow$$

in $\mathsf{K}(\mathsf{Pro}\, A)$. Hence there is a cohomology long exact sequence consisting of pieces

$$\operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\, A)}(\Sigma^i E, K) \longrightarrow \operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\, A)}(\Sigma^i E, F) \longrightarrow \operatorname{Hom}_{\mathsf{K}(\mathsf{Pro}\, A)}(\Sigma^i E, P).$$

Lemma 3.1 says that the second map here is always an isomorphism, and this implies the lemma. □

Remark 4.3. For the lemma below, recall that an augmented Gorenstein projective resolution of a module M is an exact sequence

$$\cdots \longrightarrow G^{-1} \longrightarrow G^0 \longrightarrow M \to 0$$

satisfying

(i) The modules $G^0, G^{-1}, \ldots$ are Gorenstein projective.
(ii) If the sequence is split up into short exact sequences, then in each short exact sequence the surjection is a relative epimorphism with respect to the class of Gorenstein projective modules.

The complex

$$G = \cdots \longrightarrow G^{-1} \longrightarrow G^0 \longrightarrow 0 \longrightarrow \cdots$$

is then called a Gorenstein projective resolution of M. Note that some authors call this a *proper* Gorenstein projective resolution.

Remark 4.4. Consider the short exact sequence from construction 4.1. The complex F is in $\mathsf{E}(A)$. In particular it is exact, and therefore the cohomology long exact sequence shows

$$\mathrm{H}^i K = \begin{cases} M & \text{for } i = 1, \\ 0 & \text{for } i \neq 1. \end{cases}$$

Hence there is an exact sequence

$$\cdots \longrightarrow K^{-2} \longrightarrow K^{-1} \longrightarrow K^0 \longrightarrow \operatorname{Ker} \partial_K^1 \xrightarrow{u} M \to 0.$$

Lemma 4.5. *Let M be an A-left-module, and consider the exact sequence from remark 4.4. This is an augmented Gorenstein projective resolution of M.*

Proof. I must prove that the sequence satisfies (i) and (ii) from remark 4.3.

(i) The modules $K^0, K^{-1}, \ldots$ are projective and hence Gorensten projective.

As for $\operatorname{Ker} \partial_K^1$, observe that in the short exact sequence from construction 4.1, the complex P is concentrated in non-positive cohomological degrees, so the modules P^1 and P^2 are zero. So in degrees 1

and 2, the short exact sequence gives

$$\begin{tikzcd}
K^2 \arrow[r, "\cong"] & F^2 \\
K^1 \arrow[u, "\partial_K^1"] \arrow[r, "\cong"] & F^1. \arrow[u, "\partial_F^1"']
\end{tikzcd}$$

Hence

$$\operatorname{Ker} \partial_K^1 \cong \operatorname{Ker} \partial_F^1,$$

and this last module is Gorenstein projective because the complex F is in $\mathsf{E}(A)$.

(ii) When splitting up the exact sequence from remark 4.4 into short exact sequences, each sequence but one is equal to a short exact sequence which results from splitting up K itself. Consider such a sequence

$$0 \to \operatorname{Ker} \partial_K^i \longrightarrow K^i \xrightarrow{k^i} \operatorname{Im} \partial_K^i \to 0$$

with $i \leq 0$.

Lemma 4.2 gives that lemma 3.3 applies to this situation if I set the homomorphism $K^i \xrightarrow{s} N$ from lemma 3.3 equal to $K^i \xrightarrow{k^i} \operatorname{Im} \partial_K^i$. So k^i is a relative epimorphism with respect to the class of Gorenstein projective modules.

The one short exact sequence which is missing is

$$0 \to \operatorname{Ker} u \longrightarrow \operatorname{Ker} \partial_K^1 \xrightarrow{u} M \to 0.$$

Here lemma 4.2 gives that lemma 3.4 applies if I set the homomorphism $\operatorname{Ker} \partial_K^i \xrightarrow{t} N$ from lemma 3.4 equal to $\operatorname{Ker} \partial_K^1 \xrightarrow{u} M$. So u is a relative epimorphism with respect to the class of Gorenstein projective modules. □

Remark 4.6. For the next theorem, recall that a Gorenstein projective precover of an A-left-module M is a homomorphism $G \longrightarrow M$ which is a relative epimorphism with respect to the class of Gorenstein projective modules, where G is Gorenstein projective (cf. remark 3.2).

In [1, thm. 3.1 and lem. 4.1] and [3, thm. (2.10)] is proved the following "approximation" result: Each module with finite Gorenstein projective dimension has a Gorenstein projective precover whose kernel has finite projective dimension.

On the other hand, the existence of such a precover forces a module to have finite Gorenstein projective dimension, so such precovers cannot exist more generally.

However, one salient feature of modules of finite projective dimension is that their projective resolutions are also Gorenstein projective resolutions. So an alternative view of [1, thm. 3.1 and lem. 4.1] and [3, thm. (2.10)] is that they prove that each module with finite Gorenstein projective dimension has a Gorenstein projective precover whose kernel is nice, namely, each projective resolution of the kernel is also a Gorenstein projective resolution.

This result generalizes to all modules as follows. Note again that some authors refer to my Gorenstein projective resolutions as "proper", cf. remark 4.3.

Theorem 4.7. *Let M be an A-left-module. Then there is a Gorenstein projective precover $G \longrightarrow M$ whose kernel Z satisfies that each projective resolution of Z is also a Gorenstein projective resolution of Z.*

Proof. Since all projective resolutions of Z are isomorphic in $\mathsf{K}(\mathsf{Pro}\,A)$, it is not hard to see that it is sufficient to produce a Gorenstein projective precover whose kernel Z has *one* projective resolution which is also a Gorenstein projective resolution.

However, the exact sequence from remark 4.4 and lemma 4.5 splits into exact sequences

$$0 \rightarrow \operatorname{Ker} u \longrightarrow \operatorname{Ker} \partial_K^1 \xrightarrow{u} M \rightarrow 0$$

and

$$\cdots \longrightarrow K^{-2} \longrightarrow K^{-1} \longrightarrow K^0 \longrightarrow \operatorname{Ker} u \rightarrow 0. \tag{5}$$

Lemma 4.5 implies that $\operatorname{Ker} \partial_K^1 \xrightarrow{u} M$ is a Gorenstein projective precover, and that the sequence (5) is an augmented Gorenstein projective resolution of the kernel $\operatorname{Ker} u$. But (5) is clearly also an augmented projective resolution of $\operatorname{Ker} u$ because the modules K^i are projective, and so the theorem follows with $G \longrightarrow M$ equal to $\operatorname{Ker} \partial_K^1 \xrightarrow{u} M$ and Z equal to $\operatorname{Ker} u$. □

5. Comparing absolute, relative, and Tate cohomology

This section proves that the new definition of Tate Ext groups of this paper fits into the long exact sequence (2) from the introduction (theorem 5.3).

Remark 5.1. In this section appear the relative Ext functors $\operatorname{Ext}_{\mathcal{G}}$. They are defined by

$$\operatorname{Ext}_{\mathcal{G}}^i(M, -) = \mathrm{H}^i \operatorname{Hom}_A(G, -)$$

where G is a Gorenstein projective resolution of M (cf. remark 4.3). Such a resolution exists by lemma 4.5. See [1] or [2] for an exposition of the theory of $\mathrm{Ext}^i_{\mathcal{G}}$.

Construction 5.2. Consider the short exact sequence from construction 4.1,

$$0 \to K \longrightarrow F \longrightarrow P \to 0.$$

Truncating the complexes K and F gives a new short exact sequence of complexes,

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & \operatorname{Ker}\partial^1_K \arrow[r] \arrow[u] & \operatorname{Ker}\partial^1_F \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & K^0 \arrow[r] \arrow[u] & F^0 \arrow[r] \arrow[u] & P^0 \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & K^{-1} \arrow[r] \arrow[u] & F^{-1} \arrow[r] \arrow[u] & P^{-1} \arrow[r] \arrow[u] & 0, \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

which I will denote

$$0 \to K' \longrightarrow F' \longrightarrow P \to 0. \tag{6}$$

Theorem 5.3. *Let M and N be A-left-modules. Then there is a long exact sequence*

$$\begin{aligned}
0 \quad \longrightarrow \quad & \mathrm{Ext}^1_{\mathcal{G}}(M,N) \longrightarrow \mathrm{Ext}^1(M,N) \longrightarrow \widehat{\mathrm{Ext}}^1(M,N) \\
\longrightarrow \quad & \cdots \\
\longrightarrow \quad & \mathrm{Ext}^i_{\mathcal{G}}(M,N) \longrightarrow \mathrm{Ext}^i(M,N) \longrightarrow \widehat{\mathrm{Ext}}^i(M,N) \longrightarrow \cdots,
\end{aligned}$$

natural in M and N.

Proof. Consider the short exact sequence (6) from construction 5.2. Let me set

$$P = \operatorname{res} M,$$

where $\operatorname{res} M$ is the projective resolution depending functorially on M. Recall that F' is a truncation of

$$F = e_* e^! P = e^! \operatorname{res} M.$$

Since $\operatorname{res} M$ consists of projective modules, the short exact sequence (6) is semi-split and therefore stays exact under the functor $\operatorname{Hom}_A(-, N)$. So there is a short exact sequence of complexes

$$0 \to \operatorname{Hom}_A(\operatorname{res} M, N) \longrightarrow \operatorname{Hom}_A(F', N) \longrightarrow \operatorname{Hom}_A(K', N) \to 0. \quad (7)$$

Since $\operatorname{res} M$ is a projective resolution of M, I have

$$\mathrm{H}^i \operatorname{Hom}_A(\operatorname{res} M, N) = \operatorname{Ext}^i(M, N)$$

for each i. Moreover, the form of the complex F' makes it clear that $\mathrm{H}^0 \operatorname{Hom}_A(F', N) = 0$ and that

$$\begin{aligned} \mathrm{H}^i \operatorname{Hom}_A(F', N) &= \mathrm{H}^i \operatorname{Hom}_A(F, N) \\ &= \mathrm{H}^i \operatorname{Hom}_A(e^! \operatorname{res} M, N) \\ &= \widehat{\operatorname{Ext}}^i(M, N) \end{aligned}$$

for $i \geq 1$.

Finally, lemma 4.5 says that K' is a Gorenstein projective resolution of M, shifted by one. Hence

$$\mathrm{H}^i \operatorname{Hom}_A(K', N) = \operatorname{Ext}_{\mathcal{G}}^{i+1}(M, N)$$

for $i \geq -1$. So looking at the cohomology long exact sequence of (7), starting with $\mathrm{H}^0 \operatorname{Hom}_A(F', N) = 0$, gives

$$0 \to \operatorname{Ext}_{\mathcal{G}}^1(M, N) \longrightarrow \operatorname{Ext}^1(M, N) \longrightarrow \widehat{\operatorname{Ext}}^1(M, N) \longrightarrow \cdots,$$

that is, the sequence from the theorem. □

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Department of Pure Mathematics, University of Leeds, Leeds LS2 9JT, United Kingdom

E-mail address: popjoerg@maths.leeds.ac.uk, www.maths.leeds.ac.uk/~popjoerg